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ON CONSERVATION THEOREMS, LAGRANGIAN FUNCTIONS AND VARIATIONAL PRINCIPLES FOR LINEAR CONTINUUM MECHANICS

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Abstract—It is shown that a general form of the bilinear functional in conjunction with classical concepts of thermodynamics leads to the results not admitted by the particular form commonly used in the literature. The conservation theorem for a wide class of continuous bodies is established. A generalized Lagrangian function for linear dissipative materials is introduced. Two specific examples: problem of elastodynamics and problem of thermoelasticity are selected in order to illustrate the proposed approach.

1. INTRODUCTION

The paper is an attempt to provide a uniform mathematical basis for construction of the conservation theorems for a wide class of problems of continuum mechanics. The motivation for studying the conservation theorems is their direct application to the theory of defects and in particular to the fracture mechanics. A brief account of the progress in this field was presented by G. Herrmann[1] and A. Golebiewska-Herrmann[2].

A primary interest of the author was to investigate the conservation theorem constructed formally for thermoelastic body by G. Herrmann[1]. The construction was possible due to the convolution with respect to time which replaced the classical multiplication of the thermodynamical flux and force. The idea of such convolution bilinear functional, primarily introduced by Schapery[3] and later independently by Gurtin[4] appeared to be a breakthrough in the search for variational principles in continuum mechanics. An account of the development in this search, indicating the contributions of Parkus, Biot, G. Herrmann, Ben-Amoz, Nickell, Sackman, Rafalski, was presented by Parkus[5]. Further investigations of the convolution bilinear functionals resulted in the construction of the minimum principles for heat conduction, thermoelasticity and viscoelasticity by Rafalski and later independently by Reiss and Haug (see [6]).

The usefulness of the convolution bilinear functional in G. Herrmann's construction of the conservation theorem encouraged the author to propose a more general approach to the problem. Namely, in the present work a class of bilinear functionals, which can be used to construct the conservation theorem, is introduced. For every functional from this class the corresponding thermodynamic flux and thermodynamic force are defined.

It should be emphasized that the basic concepts of thermodynamics are used here in conjunction with an appropriate class of bilinear functionals, which does not necessarily contain the classical bilinear functional. This class is determined by the type of the continuum mechanics problem. Consequently the proposed approach provides us with a set of distinct definitions of the thermodynamic flux and force for a specified problem of continuum mechanics.

It follows from Section 2 of the present work that the specification of the constitutive law, i.e. the relation between the thermodynamic flux and the thermodynamic force, is not necessary to construct the conservation theorem. However one can use the constitutive law to obtain a more convenient form of the conservation theorem, expressed in terms of the displacement function only. The construction of such conservation law has been already given by A. Golebiewska-Herrmann[2] for the materials which admit a Lagrangian. Actually, this important work stimulated the author to investigate the relations between the Lagrangian function, the

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variational principles and the conservation theorems. The original idea of the differentiation of the Lagrangian function with respect to the space and time coordinates proposed in [2] to obtain the conservation theorem is used here in the form of a skew-symmetric operator M.

A general form of the bilinear functional introduced in the present work in conjunction with the classical concept of the Lagrangian leads to the notion of the Lagrangian-like function. Now the material behavior within the considered space-time interval is uniquely determined by the pair: the particular bilinear functional and the corresponding Lagrangian-like function. Consequently for given material behavior we have, in general, several distinct Lagrangian-like functions corresponding to distinct bilinear functionals.

The Lagrangian-like function generates directly a variational principle equivalent to the appropriate boundary value problem for the considered body. It is interesting that the choice of the bilinear functional determines the type of the initial-boundary conditions appearing in the boundary value problem.

In order to illustrate the new approach proposed in the work we present four distinct Lagrangian-like functions for the problem of elastodynamics, which admits classical Lagrangian function, and one particular Lagrangian-like function for the problem of thermoelasticity, which does not admit a Lagrangian.

2. BASIC CONSERVATION THEOREM

We consider a continuous body which occupies sufficiently regular region V within the time interval $[0,t_0]$ We denote by $\varphi(\mathbf{x}, t)$ the generalized *displacement* function defined in the space-time region $V\times[0,t_0]$. We use the concept of the *thermodynamic flux* U determined by the displacement function with the relation

$$U(\mathbf{x}, t) = Q\varphi(\mathbf{x}, t) \tag{1}$$

where Q is a linear differential operator, and the concept of the *thermodynamic force* T conjugated to the flux U with a *bilinear form* $[U\circ T]$. The bilinear form introduced above is assumed to map a pair of functions $U(\mathbf{x}, t)$, $T(\mathbf{x}, t)$ into a scalar function $[U\circ T](\mathbf{x}, t)$ defined in the region $V\times[0,t_0]$. The bilinear form determines the *bilinear functional*

$$(U, T) = \int_V\int_0^{t_0}[U\circ T](\mathbf{x}, t)\,\mathrm{d}t\,\mathrm{d}V \tag{2}$$

which maps this pair of functions into a real number.

The operator Q and the bilinear functional uniquely determine the *adjoint operator* Q^*, the *boundary operator* $Q^B=[Q^{B1}, Q^{B2}]$ and the *initial operator* $Q^I=[Q^{I1}, Q^{I2}]$. The definition of these operators follows from the identity obtained with the appropriate integration by parts of the bilinear functional (W_ϕ, T). The identity takes the form

$$(Q_\phi, T)-(\varphi, Q^*T) = \int_B\int_0^{t_0}\mathbf{n}Q^B[\varphi\circ T](\mathbf{x}, t)\,\mathrm{d}t\,\mathrm{d}B + \int_V |Q^I[\varphi\circ T](\mathbf{x}, t)|_0^{t_0}\,\mathrm{d}V \tag{3}$$

where B is the boundary of the region V and $\mathbf{n}=[n_1, n_2, n_3]$ is the unit vector normal to B and taken as positive outwardly. Here we introduced the following notation

$$Q^B[\varphi\circ T](\mathbf{x}, t) = [Q^{B1}\varphi\circ Q^{B2}T](\mathbf{x}, t)$$

$$Q^I[\varphi\circ T](\mathbf{x}, t) = [Q^{I1}\varphi\circ Q^{I2}T](\mathbf{x}, t), \tag{4}$$

i.e. the differential operators with superscript 1 are applied to the displacement functions and the differential operators with superscript 2 are applied to the force functions.

In the present work we shall assume that the equation

$$Q^* T(\mathbf{x}, t) = 0 \text{ in } V \times [0, t_0] \tag{5}$$

expresses the *equilibrium condition* for the considered body. This assumption restricts the class of bilinear functionals which can be used in further considerations as well as the set of differential operators Q, which can determine the thermodynamic flux.

In order to construct the conservation theorem we introduce an auxiliary linear operator M, which is skew-symmetric with respect to the bilinear functional, i.e. $M^* = -M$. Using the notation introduced above we can write the following identity for the operator M

$$(MU, T) + (U, MT) = \int_B \int_0^{t_0} \mathbf{n} M^B[U \circ T](\mathbf{x}, t)\, \mathrm{d}t\, \mathrm{d}B + \int_V |M^I[U \circ T](\mathbf{x}, t)|_0^{t_0}\, \mathrm{d}V \tag{6}$$

where

$$M^B[U \circ T](\mathbf{x}, t) = [M^{B1} U \circ M^{B2} T](\mathbf{x}, t)$$
$$M^I[U \circ T](\mathbf{x}, t) = [M^{I1} U \circ M^{I2} T](\mathbf{x}, t). \tag{7}$$

We shall consider a class of skew-symmetric operators M such that the equality

$$MQ\varphi(\mathbf{x}, t) = QM\varphi(\mathbf{x}, t) \tag{8}$$

holds true for arbitrary displacement function $\varphi(\mathbf{x}, t)$.

Let φ^a denote the actual displacement function and let T^a denote the actual force function in the considered body. Since the actual force satisfies the equilibrium condition the relation (3) implies

$$(Q\varphi, T^a) = \int_B \int_0^{t_0} \mathbf{n} Q^B[\varphi \circ T^a](\mathbf{x}, t)\, \mathrm{d}t\, \mathrm{d}B + \int_V |Q^I[\varphi \circ T^a](\mathbf{x}, t)|_0^{t_0}\, \mathrm{d}V \tag{9}$$

for arbitrary displacement function $\varphi(\mathbf{x}, t)$. On the other hand taking into account that the actual flux function $U^a(\mathbf{x}, t)$ is *kinematically admissible* (i.e. it can be derived from the displacement function: $U^a = Q\varphi^a$) we obtain from (3)

$$(U^a, T) = \int_B \int_0^{t_0} \mathbf{n} Q^B[\varphi^a \circ T](\mathbf{x}, t)\, \mathrm{d}t\, \mathrm{d}B + \int_V |Q^I[\varphi^a \circ T](\mathbf{x}, t)|_0^{t_0}\, \mathrm{d}V \tag{10}$$

for arbitrary force function $T(\mathbf{x}, t)$ which satisfies the equilibrium condition $Q^* T = 0$.

Substituting the particular functions φ and T

$$\varphi(\mathbf{x}, t) = M\varphi^a(\mathbf{x}, t)$$
$$T(\mathbf{x}, t) = MT^a(\mathbf{x}, t) \tag{11}$$

into relations (9) and (10) and taking into account the properties (6) and (8) of the operator M we arrive at a general form of the *conservation theorem* which states that the actual displacement function φ^a and the actual force function T^a satisfy the equation

$$\int_B \int_0^{t_0} \mathbf{n} \{ Q^B[M\varphi^a \circ T^a + \varphi^a \circ MT^a](\mathbf{x}, t) - M^B[Q\varphi^a \circ T^a](\mathbf{x}, t) \}\, \mathrm{d}t\, \mathrm{d}B$$
$$+ \int_V |Q^I[M\varphi^a \circ T^a + \varphi^a \circ MT^a](\mathbf{x}, t) - M^I[Q\varphi^a \circ T^a](\mathbf{x}, t)|_0^{t_0}\, \mathrm{d}V = 0. \tag{12}$$

It should be noted that the conservation theorem (12) has been derived independently of the *constitutive law*, which describes the material behavior. Indeed, in the present section we have used only the kinematic relation (1) and the equilibrium condition (5). The constitutive law in the form of a relation between the force T^a and the flux U^a makes it possible to express the conservation theorem in terms of the displacement function φ^a only. Such conservation law will be constructed in the next sections of work.

3. LAGANGIAN-LIKE FUNCTION AND VARIATIONAL PRINCIPLE

The classical thermodynamics finds it convenient to assume that the relation between the force T and the flux U can be determined by the Lagrangian $L(U)$ which maps the value of the flux function at certain point $\mathbf{x}$, t of the space-time into a real number. It is assumed that the scalar function $L(U)$ is differentiable with respect to U and that the constitutive law is expressed in the form

$$T = \frac{\partial L}{\partial U} \tag{13}$$

where T is the value of the force at $\mathbf{x}$, t and the differentiation is carried out with respect to the *scalar product* $U \cdot T$. The relation (13) is equivalent to the statement that for every flux U there exists unique force T such that the equality

$$\lim_{\lambda \to 0} \frac{L(U + \lambda W) - L(U)}{\lambda} = W \cdot T \tag{14}$$

holds true for arbitrary flux W.

The classical assumptions presented above are so restrictive that for many important material models the Lagrangian $L(U)$ cannot be constructed. In particular the Langrangian does not exist for materials which dissipate the mechanical energy such as the thermoelastic material.

In the present paper we extend the class of materials which admit a Lagrangian by relaxation of the mathematical assumptions imposed in the classical approach. Namely, we assume that there exists an operator L which maps the flux function $U(\mathbf{x}, t)$ into a scalar function $[L(U)](\mathbf{x}, t)$ defined in the region $V \times [0, t_0]$, such that the force function $T(\mathbf{x}, t)$ corresponding to $U(\mathbf{x}, t)$ satisfies the equality

$$\lim_{\lambda \to 0} \frac{\mathscr{L}(U + \lambda W) - \mathscr{L}(U)}{\lambda} = (W, T) \tag{15}$$

for arbitrary flux function $W(\mathbf{x}, t)$. Here the functional $\mathscr{L}$ is defined by

$$\mathscr{L}(U) = \int_V \int_0^{t_0} [L(U)](\mathbf{x}, t)\, \mathrm{d}t\, \mathrm{d}V \tag{16}$$

and (W, T) denotes the bilinear functional (2) introduced in the previous section. To avoid possible confusion we shall call $[L(U)](\mathbf{x}, t)$ the ***Lagrangian-like function.***

In the sequel the constitutive law (15) will be presented in the form

$$T = \frac{\partial \mathscr{L}}{\partial U} \tag{17}$$

where the right hand side is the Gateaux derivative with respect to the bilinear functional (2).

The constitutive law in the form (17) leads directly to a variational principle corresponding to the appropriate boundary value problem. In order to formulate such boundary value problem we introduce the displacement function $\varphi^0(\mathbf{x}, t)$ prescribed in the region $V \times [0, t_0]$ and the corresponding force function $T^0(\mathbf{x}, t)$ satisfying the relation

$$T^0 = \left.\frac{\partial \mathscr{L}}{\partial U}\right|_{U=Q\varphi^0}. \tag{18}$$

We also introduce the space H_0 of all displacement functions $\psi(\mathbf{x}, t)$ which vanish on the boundary. Here the function $\psi(\mathbf{x}, t)$ is said to vanish on the boundary if it satisfies the equations

$$\begin{aligned} &Q^B[\psi \circ (T^0 - T)](\mathbf{x}, t) = 0 \quad \text{on } B\chi[0, t_0] \\ &Q^I[\psi \circ (T^0 - T)](\mathbf{x}, t) = 0 \quad \text{in } V \text{ for } t = 0 \text{ and } t = t_0 \end{aligned} \tag{19}$$

where the force function $T(\mathbf{x}, t)$ is determined by $\psi(\mathbf{x}, t)$ with

$$T = \left.\frac{\partial \mathscr{L}}{\partial U}\right|_{U=Q\varphi^0 - Q\psi}. \tag{20}$$

Now we can formulate the boundary value problem as follows:
Find the displacement function φ^a which satisfies the boundary conditions determined by the prescribed function φ^0, i.e. the function φ^a which can be expressed in the form

$$\varphi^a = \varphi^0 - \psi \quad \text{where } \psi \text{ belongs to } H_0 \tag{21}$$

such that the corresponding force function T^a determined by the constitutive law

$$T^a = \left.\frac{\partial \mathscr{L}}{\partial U}\right|_{U=Q\varphi^a} \tag{22}$$

satisfies the equilibrium condition

$$Q^* T^a(\mathbf{x}, t) = 0 \text{ in } V \times (0, t_0]. \tag{23}$$

One can show that the above problem is equivalent to the following *variational principle*:

The functional

$$F(\psi) = \mathscr{L}(Q[\varphi^0 - \psi]) + \int_B \int_0^{t_0} \mathbf{n} Q^B[\psi \circ T^0](\mathbf{x}, t)\, \mathrm{d}t\, \mathrm{d}B + \int_V |Q^I[\psi \circ T^0](\mathbf{x}, t)|_0^{t_0} \mathrm{d}V \tag{24}$$

defined in the space H_0 of all displacement functions vanishing on the boundary attains its stationary value if and only if $\psi = \varphi^0 - \varphi^a$, where φ^a is the actual displacement function.

Indeed, taking into account the relations (3), (15) and (19) we arrive at the equation

$$\lim_{\lambda \to 0} \frac{F(\psi + \lambda\varphi) - F(\psi)}{\lambda} = -(\varphi, Q^* T) \tag{25}$$

which can be presented as

$$\frac{\partial F}{\partial \psi} = -Q^* T. \tag{26}$$

Hence the requirement $(\partial F/\partial\psi) = 0$ is equivalent to the equilibrium condition $Q^* T = 0$.

The particular form of the constitutive law expressed by the Lagrangian-like function imposes the uniqueness of the force function T corresponding to given displacement function φ. Making use of this property one can formally express the force function T^a appearing in the conservation theorem (12) by the displacement function (22) in order to obtain a conservation theorem expressed in terms of the displacement only. In the case where only boundary values of the displacement function are known such substitution requires an effective solution of the boundary value problem.

It will be shown in the sequel that the particular bilinear functional introduced by Schapery[3] makes it possible to construct the Lagrangian-like function for the problem of thermoelasticity. Analogous construction is also possible for other dissipative media characterized by a linear constitutive law.

It will be also shown that for the classical problem of elastodynamics one can construct several Lagrangian-like functions corresponding to distinct bilinear functionals.

4. LINEAR CONSTITUTIVE LAW

Now we shall assume that the constitutive law is linear, i.e. the mapping of the flux function $U(\mathbf{x}, t)$ into the force function $T(\mathbf{x}, t)$ is such that

$$T(\lambda_1 U + \lambda_2 W) = \lambda_1 T(U) + \lambda_2 T(W) \tag{27}$$

for arbitrary flux functions U, W and real numbers λ_1, λ_2.

It can be shown that the functional $\mathscr{L}$ for a linear constitutive law can be constructed as

$$\mathscr{L}(U) = \frac{1}{2}(U, T(U)) \tag{28}$$

provided that the mapping $T(U)$ is *symmetric with respect to* the assumed *bilinear functional*, i.e. the equality

$$(W, T(U)) = (U, T(W)) \tag{29}$$

holds true for arbitrary flux functions U and W.

The symmetry condition (29) can be used as a criterion of usefulness of a bilinear functional to construct the Lagrangian-like function. Namely for particular linear constitutive law we restrict the class of bilinear functionals introduced above to those which make the mapping $T(U)$ symmetric (29). It follows from further considerations that the restricted class of the bilinear functionals admits a simplified construction of the conservation theorems.

It should be emphasized that the symmetry (29) of the mapping $T(U)$ with respect to the bilinear functional (2) does not imply such symmetry with respect to the bilinear form, i.e. the function $[W \circ T(U) - U \circ T(W)](\mathbf{x}, t)$ does not necessarily vanish in the region $V \times [0, t_0]$. Even for a simple form of the linear constitutive law

$$T(\mathbf{x}, t) = \mathbf{C}\, U(\mathbf{x}, t) \tag{30}$$

where $\mathbf{C}$ is a symmetric constant tensor, the symmetry (29) can be satisfied while the skew-symmetric part of the bilinear form $[W \circ \mathbf{C} U](\mathbf{x}, t)$ defined by

$$[W \circ \mathbf{C} U]_A(\mathbf{x}, t) = \frac{1}{2}\{[W \circ \mathbf{C} U](\mathbf{x}, t) - [\mathbf{C} U \circ W](\mathbf{x}, t)\} \tag{31}$$

does not vanish in $V \times [0, t_0]$. This relation follows from the property of the bilinear form which is not necessarily commutative.

The symmetry (29) of the mapping $T(U)$ simplifies the construction of a conservation theorem. Indeed, assuming that the skew-symmetric operator M satisfies the relation

$$T(MU) = MT(U) \tag{32}$$

for arbitrary flux function $U(\mathbf{x}, t)$ and taking into account the symmetry (29) and the properties

(3) of the operator Q we obtain

$$(M\varphi^a, Q^*T^a) = \frac{1}{2}\{(MQ\varphi^a, T^a) + (Q\varphi^a, MT^a)\} - \int_B \int_0^{t_0} \mathbf{n} Q^B[M\varphi^a \circ T^a](\mathbf{x}, t)\,dt\,dB - \int_V |Q^I[M\varphi^a \circ T^a](\mathbf{x}, t)|_0^{t_0}\,dV. \tag{33}$$

Using the properties (6) of the operator M and the equilibrium condition (5) we arrive at the *conservation theorem* in the form

$$\int_B \int_0^{t_0} \mathbf{n}\{M^B\left[\frac{1}{2}Q\varphi^a \circ T^a\right](\mathbf{x}, t) - Q^B[M\varphi^a \circ T^a](\mathbf{x}, t)\}\,dt\,dB + \int_V |M^I\left[\frac{1}{2}Q\varphi^a \circ T^a\right](\mathbf{x}, t) - Q^I[M\varphi^a \circ T^a](\mathbf{x}, t)|_0^{t_0}\,dV = 0 \tag{34}$$

where the actual force function T^a is uniquely determined by the actual displacement function φ^a.

If the boundary and initial operators take the particular form of the multiplication of the bilinear form by a constant vector then we can explicitly introduce the Lagrangian-like function into the conservation theorem (34)

$$\begin{aligned} M^B\left[\frac{1}{2}Q\varphi^a \circ T^a\right](\mathbf{x}, t) &= M^B[L(Q\varphi^a)](\mathbf{x}, t) \\ M^I\left[\frac{1}{2}Q\varphi^a \circ T^a\right](\mathbf{x}, t) &= M^I[L(Q\varphi^a)](\mathbf{x}, t). \end{aligned} \tag{35}$$

5. PARTICULAR BILINEAR FORMS

In order to construct the conservation theorems one can look for a skew-symmetric operator in the form

$$M = m_1 \frac{\partial^p}{\partial x_1^p} + m_2 \frac{\partial^q}{\partial x_2^q} + m_3 \frac{\partial^r}{\partial x_3^r} + m_4 \frac{\partial^s}{\partial t^s} \tag{36}$$

where $\boldsymbol{m} = [m_1, m_2, m_3, m_4]$ is interpreted as a vector in four-dimensional space-time and p, q, r, s are positive or negative odd integers (negative order of a derivative denotes appropriate integral).

In our consideration we shall use only first order derivatives, namely we shall assume $p = q = r = s = 1$. Hence the form (36) offers at most four operators M independent of each other. Now we can use the concept of the *energy-momentum function* $\mathbf{P}(\mathbf{x}, t)$ (see [1]) in order to express the *conservation theorem* (34) in the form

$$\int_B \int_0^{t_0} m_r n_j P_{rj}(\mathbf{x}, t)\,dt\,dB + \int_V |m_r P_{r4}(\mathbf{x}, t)|_0^{t_0}\,dV = 0 \tag{37}$$

$$\int_B \int_0^{t_0} m_4 n_j P_{4j}(\mathbf{x}, t)\,dt\,dB + \int_V |m_4 P_{44}(\mathbf{x}, t)|_0^{t_0}\,dV = 0 \tag{38}$$

where the summation convention is used for the indices $r, j = 1, 2, 3$.

In the sequel we shall use four particular bilinear forms to carry out the constructions proposed in previous sections. The *first bilinear form*, recommended by classical thermodynamics,

$$[U \circ T](\mathbf{x}, t) = U(\mathbf{x}, t) \cdot T(\mathbf{x}, t) \tag{39}$$

admits four skew-symmetric operators M independent of each other, i.e. the corresponding operator M is represented by four non-zero components of the vector $\mathbf{m} = [m_1, m_2, m_3, m_4]$.

The *second bilinear form*, characterized by the convolution with respect to time,

$$[U \circ T](\mathbf{x}, t) = U(\mathbf{x}, t) \cdot T(\mathbf{x}, t_0 - t) \tag{40}$$

admits three non-zero components of the vector $\mathbf{m} = [m_1, m_2, m_3, 0]$. Hence the eqn (38) becomes trivial for this form.

The *third bilinear form*, characterized by the convolution with respect to $\mathbf{x}$,

$$[U \circ T](\mathbf{x}, t) = U(\mathbf{x}, t) \cdot T(-\mathbf{x}, t) \tag{41}$$

admits one non-zero component of the vector $\mathbf{m} = [0, 0, 0, m_4]$. Now the eqn (37) becomes trivial.

The *fourth bilinear form*, characterized by the convolution both with respect to $\mathbf{x}$ and t,

$$[U \circ T](\mathbf{x}, t) = U(\mathbf{x}, t) \cdot T(-\mathbf{x}, t_0 - t) \tag{42}$$

does not admit non-zero components of the vector $\mathbf{m} = [0, 0, 0, 0]$. Hence this form does not offer any non-trivial conservation theorem generated by the considered class of operators M.

It should be noted that the third and fourth bilinear forms can be applied to a class of regions V which are symmetric with respect to three axes of the coordinate system in the space R^3.

6. ELASTODYNAMICS

The displacement function $\varphi(\mathbf{x}, t)$ is here identified with the displacement $u_i(\mathbf{x}, t)$ of the material points. For the first bilinear form (39) the flux takes the form

$$u(\mathbf{x}, t) \equiv \begin{bmatrix} u_{i,j}(\mathbf{x}, t) \\ \dot{u}_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} (\partial/\partial x_j) \\ (\partial/\partial t) \end{bmatrix} u_i(\mathbf{x}, t) \equiv Q\varphi(\mathbf{x}, t) \tag{43}$$

and the constitutive law is expressed by

$$T(\mathbf{x}, t) \equiv \begin{bmatrix} \sigma_{ij}(\mathbf{x}, t) \\ -p_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} C_{ijkl} & 0 \\ 0 & -\delta_{ik}\rho \end{bmatrix} \begin{bmatrix} u_{k,l}(\mathbf{x}, t) \\ \dot{u}_k(\mathbf{x}, t) \end{bmatrix} \equiv \mathbf{C}\, U(\mathbf{x}, t) \tag{44}$$

where σ_{ij} is the stress tensor, p_i the momentum vector, C_{ijkl} the generalized Young modulus, ρ the material density and δ_{ij} is the Kronecker symbol. Due to the symmetry of the mapping (44) (see(29)) with respect to the first bilinear form the Lagrangian-like function takes the classical form

$$[L(U)](\mathbf{x}, t) = \frac{1}{2} U(\mathbf{x}, t) \cdot \mathbf{C}\, U(\mathbf{x}, t). \tag{45}$$

The energy-momentum function appearing in the conservation theorem (37), (38) is now expressed by

$$\begin{aligned} P_{rj}(\mathbf{x}, t) &= \delta_{rj}[L(U)](\mathbf{x}, t) - [u_{i,r} \circ C_{ijkl} u_{k,l}](\mathbf{x}, t) \\ P_{r4}(\mathbf{x}, t) &= [u_{i,r} \circ \rho \dot{u}_i](\mathbf{x}, t) \\ P_{4j}(\mathbf{x}, t) &= -[u_i \circ C_{ikjl} u_{k,l}](\mathbf{x}, t) \\ P_{44}(\mathbf{x}, t) &= [L(U)](\mathbf{x}, t) + [\dot{u}_i \circ \rho \dot{u}_i](\mathbf{x}, t). \end{aligned} \tag{46}$$

The natural boundary conditions (see eqns 19) imposed by the first bilinear form state that the displacement function $u_i(\mathbf{x}, t)$ is prescribed on the boundary $B \times [0, t_0]$ as well as in the region V at the initial $t = 0$ and final $t = t_0$ moments.

For the second bilinear form (40) the flux takes the form

$$U(\mathbf{x}, t) \equiv \begin{bmatrix} u_{i,j}(\mathbf{x}, t) \\ u_i(\mathbf{x}, t) \\ \ddot{u}_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} (\partial/\partial x_j) \\ 1 \\ (\partial^2/\partial t^2) \end{bmatrix} u_i(\mathbf{x}, t) \equiv Q\varphi(\mathbf{x}, t) \tag{47}$$

and the constitutive law is presented as

$$T(\mathbf{x},t) \equiv \begin{bmatrix} \sigma_{ij}(\mathbf{x},t) \\ \frac{1}{2}\ddot{f}_i(\mathbf{x},t) \\ \frac{1}{2}f_i(\mathbf{x},t) \end{bmatrix} = \begin{bmatrix} C_{ijkl} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\delta_{ik}\rho \\ 0 & \frac{1}{2}\delta_{ik}\rho & 0 \end{bmatrix} \begin{bmatrix} u_{k,l}(\mathbf{x},t) \\ u_k(\mathbf{x},t) \\ \ddot{u}_k(\mathbf{x},t) \end{bmatrix} \equiv \mathbf{C}U(\mathbf{x},t) \tag{48}$$

where $\ddot{f}_i(\mathbf{x}, t)$ is the inertia force.

The Lagrangian-like function corresponding to the second bilinear form can be expressed by

$$[L(U)](\mathbf{x},t) = \frac{1}{2}U(\mathbf{x},t)\cdot \mathbf{C}U(\mathbf{x},t_0-t). \tag{49}$$

The energy-momentum function, appearing in the conservation theorem (37) takes the form

$$\begin{aligned} P_{rj}(\mathbf{x},t) &= \delta_{rj}[L(U)](\mathbf{x},t) - [u_{i,r}\circ C_{ijkl}u_{k,l}](\mathbf{x},t) \\ P_{r4}(\mathbf{x},t) &= -\frac{1}{2}[u_{i,r}\circ\rho\dot{u}_i + \dot{u}_{i,r}\circ\rho u_i](\mathbf{x},t). \end{aligned} \tag{50}$$

The natural boundary conditions imposed by the second bilinear form state that the displacement function $u_i(\mathbf{x},t)$ is prescribed on the boundary $B\times[0,t_0]$ and that this function together with its time derivative is prescribed in the region V at the initial moment $t=0$.

For the *third bilinear form* (41) the flux takes the form

$$U(\mathbf{x},t) \equiv \begin{bmatrix} u_{i,jk}(\mathbf{x},t) \\ u_i(\mathbf{x},t) \\ \dot{u}_i(\mathbf{x},t) \end{bmatrix} = \begin{bmatrix} (\partial/\partial x_j)(\partial/\partial x_k) \\ 1 \\ (\partial/\partial t) \end{bmatrix} u_i(\mathbf{x},t) \equiv (Q\varphi(\mathbf{x},t) \tag{51}$$

and the constitutive law is presented by

$$T(\mathbf{x},t) \equiv \begin{bmatrix} \frac{1}{2}b_{ijk}(\mathbf{x},t) \\ \frac{1}{2}b_{ijk,jk}(\mathbf{x},t) \\ -p_i(\mathbf{x},t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}C_{ijkl} & 0 \\ \frac{1}{2}C_{ijkl} & 0 & 0 \\ 0 & 0 & -\delta_{il}\rho \end{bmatrix} \begin{bmatrix} u_{l,kj}(\mathbf{x},t) \\ u_l(\mathbf{x},t) \\ \dot{u}_l(\mathbf{x},t) \end{bmatrix} \equiv \mathbf{C}U(\mathbf{x},t) \tag{52}$$

where the tensor b_{ijk} may be interpreted as the potential of stress (since $\sigma_{ij} = b_{ijk,k}$). Consequently $b_{ijk,jk}$ represents the body force.

The Lagrangian-like function corresponding to the third bilinear form can be presented as

$$[L(U)](\mathbf{x},t) = \frac{1}{2}U(\mathbf{x},t)\mathbf{C}U(-\mathbf{x},t) \tag{53}$$

and the energy-momentum function appearing in the conservation theorem (38) is expressed by

$$\begin{aligned} P_{4j}(\mathbf{x},t) &= \frac{1}{2}[\dot{u}_{m,n}\circ C_{mnkj}u_k + \dot{u}_i\circ C_{ijkl}u_{k,l}](\mathbf{x},t) \\ P_{44}(\mathbf{x},t) &= -[L(U)](\mathbf{x},t). \end{aligned} \tag{54}$$

In order to specify the boundary conditions for the third bilinear form we decompose the boundary B into two disjoint surfaces B_1 and B_2 such that for arbitrary x belonging to B_1 we have: $-\mathbf{x}$ belongs to B_2. Now we can propose a natural boundary condition in the form: $u_i(\mathbf{x}, 0)$ and $u_i(\mathbf{x}, t_0)$ are prescribed in V and $u_i(\mathbf{x}, t)$ together with $n_k u_{i,k}(\mathbf{x}, t)$ are prescribed on the surface $B_1 \times [0, t_0]$.

For the *fourth bilinear form* (42) the flux takes the form

$$U(\mathbf{x}, t) \equiv \begin{bmatrix} u_{i,jk}(\mathbf{x}, t) \\ u_i(\mathbf{x}, t) \\ \ddot{u}_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} (\partial/\partial x_j)\,(\partial/\partial x_k) \\ 1 \\ (\partial^2/\partial t^2) \end{bmatrix} u_i(\mathbf{x}, t) \equiv Q\varphi(\mathbf{x}, t) \tag{55}$$

and the constitutive law is presented by

$$T(\mathbf{x}, t) \equiv \begin{bmatrix} \frac{1}{2} b_{ijk}(\mathbf{x}, t) \\ \frac{1}{2}\{b_{ijk,jk} + \ddot{f}_i\}(\mathbf{x}, t) \\ \frac{1}{2} f_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} C_{ijkl} & 0 \\ \frac{1}{2} C_{ijkl} & 0 & \frac{1}{2}\delta_{il}\rho \\ 0 & \frac{1}{2}\delta_{il}\rho & 0 \end{bmatrix} \begin{bmatrix} u_{l,kj}(\mathbf{x}, t) \\ u_l(\mathbf{x}, t) \\ \ddot{u}_l(\mathbf{x}, t) \end{bmatrix} \equiv \mathbf{C}\, U(\mathbf{x}, t). \tag{56}$$

The boundary conditions for the fourth bilinear form can be established as follows: the functions $u_i(\mathbf{x}, 0)$ and $\dot{u}_i(\mathbf{x}, 0)$ are prescribed in V and the functions $u_i(\mathbf{x}, t)$ and $n_k u_{i,k}(\mathbf{x}, t)$ are prescribed on $B_1 \times [0, t]$.

7. THERMOELASTICITY

The displacement function $\varphi(\mathbf{x}, t)$ is here composed of the displacement of the material $u_i(\mathbf{x}, t)$ and the entropy displacement $s_i(\mathbf{x}, t)$. Applying the second bilinear form we can express the flux as

$$U(\mathbf{x}, t) \equiv \begin{bmatrix} u_{i,j}(\mathbf{x}, t) \\ s_{k,k}(\mathbf{x}, t) \\ s_i(\mathbf{x}, t) \\ \dot{s}_i(\mathbf{x}, t) \\ u_i(\mathbf{x}, t) \\ \ddot{u}_l(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} \delta_{ik}\delta_{jl}(\partial/\partial x_l) & 0 \\ 0 & (\partial/\partial x_k) \\ 0 & \delta_{ik} \\ 0 & \delta_{ik}(\partial/\partial t) \\ \delta_{ik} & 0 \\ \delta_{ik}(\partial^2/\partial t^2) & 0 \end{bmatrix} \begin{bmatrix} u_k(\mathbf{x}, t) \\ s_k(\mathbf{x}, t) \end{bmatrix} \equiv Q\varphi(\mathbf{x}, t). \tag{57}$$

The corresponding constitutive law takes the form

$$\begin{bmatrix} \sigma_{ij}(\mathbf{x}, t) \\ -\theta(\mathbf{x}, t) \\ \frac{1}{2}\dot{g}_i(\mathbf{x}, t) \\ \frac{1}{2} g_i(\mathbf{x}, t) \\ \frac{1}{2}\ddot{f}_i(\mathbf{x}, t) \\ \frac{1}{2} f_i(\mathbf{x}, t) \end{bmatrix} = \begin{bmatrix} C'_{ijkl} & \eta\beta_{ij} & 0 & 0 & 0 & 0 \\ \eta\beta_{kl} & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\chi_{ik} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\chi_{ik} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\delta_{ik}\rho \\ 0 & 0 & 0 & 0 & \frac{1}{2}\delta_{ik}\rho & 0 \end{bmatrix} \begin{bmatrix} u_{k,l}(\mathbf{x}, t) \\ s_{k,k}(\mathbf{x}, t) \\ s_k(\mathbf{x}, t) \\ \dot{s}_k(\mathbf{x}, t) \\ u_k(\mathbf{x}, t) \\ \ddot{u}_l(\mathbf{x}, t) \end{bmatrix} \tag{58}$$

where θ is the increment of the temperature above the reference temperature θ_0, $\dot{g}_i$ is the temperature gradient, β_{ij} is the coefficient related to thermal expansion, k_{ij} is the coefficient of heat conduction, c is the heat capacity and

$$\eta = \frac{\theta_0}{c}, \quad C'_{ijkl} = C_{ijkl} + \eta\beta_{ij}\beta_{kl}, \quad \chi_{ik} = \frac{\theta_0}{k_{ik}}. \tag{59}$$

According to the notation used throughout the work we express the constitutive law (58) in the form $T(\mathbf{x}, t) = \mathbf{C}\,U(\mathbf{x}, t)$. Then the Lagrangian-like function for the thermoelastic problem can be presented in the form

$$[L(U)](\mathbf{x}, t) = \frac{1}{2}\,U(\mathbf{x}, t)\cdot \mathbf{C}\,U(\mathbf{x}, t_0 - t). \tag{60}$$

The corresponding energy-momentum function appearing in the conservation theorem (37) can be expressed by

$$\begin{aligned} P_{rj}(\mathbf{x}, t) &= \delta_{rj}[L(U)](\mathbf{x}, t) - [u_{i,r}\circ\sigma_{ij} - s_{k,k}\circ\theta](\mathbf{x}, t) \\ P_{r4}(\mathbf{x}, t) &= -\frac{1}{2}[u_{i,r}\circ\rho\dot{u}_i + \dot{u}_{i,r}\circ\rho u_i + s_{i,r}\circ\chi_{ij}s_j](\mathbf{x}, t) \end{aligned} \tag{61}$$

where the stress σ_{ij} and the temperature increment θ are determined by the flux U with the equality (58).

The natural boundary conditions corresponding to the second bilinear form (40) are: the functions $u_i(\mathbf{x}, 0)$, $\dot{u}_i(\mathbf{x}, 0)$ and $s_i(\mathbf{x}, 0)$ are prescribed in the region V and the functions $u_i(\mathbf{x}, t)$ and $s_i(\mathbf{x}, t)$ are prescribed on the boundary $B \times [0, t_0]$.

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